



**SB-3531**  
**M. Sc. - II Examination**  
**March / April - 2011**  
**Abstract Algebra**  
**(New Course)**

Time :        Hours]

[Total Marks :

**Instructions :**

(1)

<p>नीचे दृष्टावेक निशानीवाणी विगतो उत्तरवही पर अवश्य लभवी. Fillup strictly the details of signs on your answer book.</p> <p>Name of the Examination :</p> <p>☛ <b>M. SC. - 2</b></p> <p>Name of the Subject :</p> <p>☛ <b>ABSTRACT ALGEBRA (NEW)</b></p> <p>☛ Subject Code No. : <b>3 5 3 1</b>    Section No. (1, 2,.....) : <b>Nil</b></p>	<p>Seat No. :</p> <table border="1" style="width: 100%; height: 20px;"><tr><td style="width: 15%;"></td><td style="width: 15%;"></td><td style="width: 15%;"></td><td style="width: 15%;"></td><td style="width: 15%;"></td><td style="width: 15%;"></td></tr></table> <div style="border: 1px solid black; border-radius: 15px; height: 60px; margin-top: 10px; display: flex; align-items: center; justify-content: center; padding: 10px;">Student's Signature</div>						

- (2) Answer all questions.  
(3) All questions carry **equal** marks.  
(4) Follow usual notations.

- 1 (a) If  $p$  is a prime number and  $p^\alpha | 0(G)$  then prove 5  
that  $G$  has subgroup of order  $p^\alpha$ .
- (b) Prove that an abelian group  $G$  has a composition 5  
series if and only if  $G$  is finite.
- (c) Prove that a group  $G$  is solvable if and only if  $G$  4  
has a normal series with abelian factors.

**OR**

- 1 (a) If  $p$  is a prime number and  $P | 0(G)$  then prove 5  
that  $G$  has an element of order  $p$ .
- (b) Prove that any group of order  $11^2 \cdot 13^2$  is abelian. 5
- (c) Prove that any two composition series of a finite 4  
group are equivalent.
- 2 (a) If  $(N_i), l \leq i \leq k$  is a family of R-submodules of a 5  
module  $M$ , then prove that

$$\sum_{i=1}^k N_i = \{x_1 + \dots + x_k \mid x_i \in N_i\}$$

- (b) Let  $M$  be a finitely generated free module over a commutative ring  $R$  then prove that all bases of  $M$  have the same number of elements. 5
- (c) Let  $R$  be a ring with unity. Let  $Hom_R(R, R)$  denote the ring of endomorphisms of  $R$  regarded as a right  $R$ -module. prove that  $R \simeq Hom_R(R, R)$  as a rings. 4

**OR**

- 2 (a) Let  $R$  be a ring with unity and let  $M$  be an  $R$ -module than prove that  $M$  is simple if and only if  $M \simeq R/I$  where  $I$  is a maximal left ideal of  $R$ . 5
- (b) For any two idempotents  $e, f$  in  $R$ , prove that  $Re + Rf = Re \oplus R(f - fe)$ . 5
- (c) Let  $A$  and  $B$  be  $R$ -submodules of  $R$ -modules  $M$  and  $N$  respectively. Prove that  $\frac{M \times N}{A \times B} \simeq \frac{M}{A} \times \frac{N}{B}$ . 4

- 3 (a) Prove that if every submodule of an  $R$ -module  $M$  is finitely generated then every nonempty set  $S$  of submodules of  $M$  has a maximal element. 5
- (b) Let  $A$  be a minimal left ideal in a ring  $R$ . Prove that either  $A^2 = (0)$  or  $A = Re$  where  $e$  is an idempotent in  $R$ . 5
- (c) Find the invariant factors of the matrix 4

$$\begin{bmatrix} -x & 4 & -2 \\ -3 & 8-x & 3 \\ 4 & -8 & -2-x \end{bmatrix} \text{ over the ring } Q[x].$$

**OR**

- 3 (a) Let  $R$  be a noetherian ring having no nonzero nilpotent ideals. Prove that  $R$  has no nonzero nil ideals. 5
- (b) Prove that every submodule and every homomorphic image of a noetherian module is noetherian. 5
- (c) Prove that the multiplicative group of non-zero elements of a finite field is cyclic. 4
- 4 (a) Define a nilpotent group. Prove that a group of order  $p^n$  ( $p$  prime) is nilpotent. 5

- (b) Prove that every finite extension of a field is an algebraic extension but converse is not true. 5
- (c) For a field  $Q$  of rational numbers, prove that 4
- $$Q(\sqrt{2}, \sqrt[3]{5}) = Q(\sqrt{2}, +\sqrt[3]{5}).$$

**OR**

- 4 (a) If  $P(x)$  is a polynomial in  $F[x]$  of degree  $n \geq 1$  and is irreducible over  $F$  then prove that there is an extension field  $E$  of  $F$  such that  $[E:F]$ , in which  $P(x)$  has a root. 5
- (b) Let  $F$  be the field of rational numbers. Find the splitting field for the polynomial  $x^4 - 2$  over  $F$ . 5
- (c) Show that it is not possible by straightedge and compass alone to trisect  $60^\circ$ . 4
- 5 (a) If  $F$  is a field of characteristic zero and  $a, b$  are algebraic over  $F$  then prove that there exists an element  $c \in F(a, b)$  such that  $F(a, b) = F(c)$ . 5
- (b) Construct the Galois group for the polynomial  $f(x) = x^3 - 2$  over  $Q$ . 5
- (c) Suppose that the field  $F$  contains all the  $n^{\text{th}}$  roots of unity and  $0 \neq a \in F$ . Let  $x^n - a \in F[x]$  and  $K$  be its splitting field over  $F$  then prove that 4
- (i)  $K = F(u)$  where  $u$  is any root of  $x^n - a$ .
- (ii) The Galois group of  $x^n - a$  over  $F$  is abelian.

**OR**

- 5 (a) Let  $F$  be a field of characteristic zero contains all the  $n^{\text{th}}$  roots of unity for every integer  $n$ . If  $P(x) \in F[x]$  is solvable by radicals over  $F$  then prove that Galois group over  $F$  of  $P(x)$  is a solvable group. 5
- (b) Prove that the field of complex numbers is a normal extension of the field of real numbers. 5
- (c) Prove that algebraic closures of a field  $F$  are unique upto isomorphism. 4